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REGULAR SUBMODELS OF TYPES (1,2) AND (1,1) OF THE EQUATIONS OF GAS DYNAMICS

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Partially invariant solutions of types $(1,2)$ *and* $(1,1)$ *for gas-dynamic equations are regularly divided into two classes: for the first class, the invariant independent variable is the time, i.e., this class contains barochronic solutions, and for the second class, the invariant variable necessarily depends on spatial coordinates. The barochronic submodel of gas-dynamic equations, as well as a passive subsystem for solutions of the second class, is integrated in finite form. In the latter case, the invariant subsystem is reduced to an ordinary differential equation and quadratures. Integration of the submodels is illustrated by a number of examples. The following common properties of barochronic gas flows are described: rectilinear trajectories of gas particles, the possibility of collapse of density on a manifold, and stratification of the space of events.*

Introduction. Partially invariant solutions (PIS) [1] have been widely used in different fields of mechanics and mathematical physics. They are a more general object compared to invariant solutions, but search for them involves great difficulties. It is possible to distinguish a class of regular partially invariant solutions (RPIS) [2], which, having the generality of PIS, can be described fairly simply. In [3], all 100 representatives of RPIS for the equations of gas dynamics (EGD) with an arbitrary equation of state are listed and individual classes of RPIS are described. Twelve RPIS of type (2,1) are described in [4]. The interesting and informative class of barochronic solutions of the EGD is dealt with in [5, 6].

In the present paper, we consider the general method of studying RPIS of types (1,2) and (1,1) using particular examples. Since a detailed description of these solutions is cumbersome, it will be given elsewhere. The analytical description of RPIS for the EGD with the equation of state of a general form is thus completed.

1. Common Properties of Barochronic Solutions. Solutions of EGD for which the pressure is a function of only time [$p = p(t)$] are called *barochronic* [3]. Barochronic solutions for which density is also a function of only time [$\rho = \rho(t)$] are RPIS of type (1,3). They correspond to isentropic gas flows. Special barochronic RPIS are solutions of types (1,2) and (1,1), and in the case of invariant solutions, they are of type (1,0).

Barochronic solutions describe inertial gas flows, which are analogs of elementary mechanical motion. They are of great interest as a source of nontrivial mathematical problems combining the theory of mapping, geometry, and algebra. At the same time, the physics of the phenomena described by these solutions is informative and rather complex. Thus, for example, studies of the existence and behavior of the general solution for a continuous medium without pressure leads to measure-valued solutions [7, 8].

Solutions of the EGD for barochronic motion can be generally written as finite formulas [5, 6]. The general solution formulas specify the velocity components $\boldsymbol{u}$ as implicit functions of all independent variables t and $\boldsymbol{x}$, and they contain arbitrary functions, whose number depends on the algebraic structure of the Jacobian matrix $J = \partial \boldsymbol{u}/\partial \boldsymbol{x}$. Density is obtained explicitly as a rational function of time.

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For barochronic motion, the initial velocity field $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$ has a special form: the algebraic invariants of its Jacobian matrix $J_0 = \partial \boldsymbol{u}_0/\partial \boldsymbol{x}$ are constant numbers. A detailed description of such vector fields of arbitrary dimension is given in terms of the systems defining $\boldsymbol{u}_0$ as an implicit function.

The solution $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is derived from the initial field $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$ by a simple substitution.

The approach based on the special properties of the matrix J can also be used to integrate the other classes of RPIS. In this case, the determining factor is that the eigenvalues J depend on just one invariant independent variable. This approach is demonstrated below for two submodels.

Barochronic gas flows have the following common properties:

1. The trajectories of gas particles in barochronic motion are *straight lines*. Each particle moves in a straight line whose position in space and velocity are determined by the initial data.

2. For all cases of three-dimensional barochronic motion and for some cases of two-dimensional barochronic motion, the solution formulas describe *the collapse of density* at the final time: at $t = t_*$ and $\rho = \infty$ on a certain manifold, whose dimension is lower than the dimension of the motion (surface, line, and point).

The behavior of the gas near the manifold of the collapse and the continuation of the solution behind the collapse have not been adequately studied.

3. The space of events is separated into *strata-manifolds*, which are mapped onto the collapse manifold in an irregular manner (with decrease in dimension).

These properties are derived from the general solution formulas for barochronic motion [5]. We illustrate them by two examples, which also show the greater generality of PIS compared to invariant solutions of the same rank: they are determined with functional arbitrariness.

Example 1. An invariant [type (1,0)] barochronic solution with respect to the subalgebra $L_3 = \{t\partial_y + \partial_v, t\partial_z + \partial_w, t\partial_x + \partial_u + k(y\partial_z - z\partial_v + v\partial_w - w\partial_v)\}$, where k is a real parameter, is specified by the formulas

$$u = \frac{x}{t}, \quad v = \frac{y + b\cos\varphi}{t}, \quad w = \frac{z + b\sin\varphi}{t}, \quad \varphi = \frac{kx}{t}, \quad \rho = \frac{\rho_0}{t^3}, \quad \rho_0, b = \text{const}. \tag{1.1}$$

The equations of trajectories have the form

$$x = t\xi, \qquad y = t\eta + (t-1)b\cos k\xi, \quad z = t\zeta + (t-1)b\sin k\xi,$$

where $(x, y, z)\big|_{t=1} = (\xi, \eta, \zeta)$.

Collapse of density occurs at $t_* = 0$ in the plane $x = 0$, and the particles come to the circle

$$y^2 + z^2 = b^2 \tag{1.2}$$

in this plane. Particles from the planes $\xi = \text{const}$ arrive at each point (y, z) of the circle (1.2), and for $\xi = \xi_0$ and $\xi = \xi_0 + 2n\pi/k$ $(n = 1, 2, \ldots)$, they arrive at the same point (y, z).

Example 2. RPIS of type (1,1) with respect to the subalgebra $L_4 = \{\partial_x, t\partial_x + \partial_u, t\partial_y + \partial_v, t\partial_z + \partial_w\}$ have the form

$$u = \frac{Cx + U(y/t, z/t)}{1 + C(t-1)}, \quad v = \frac{y}{t}, \quad w = \frac{z}{t}, \quad \rho = \frac{\rho_0}{t^2(1 + C(t-1))}, \quad C, \rho_0 = \text{const}, \tag{1.3}$$

where U is an arbitrary function of its arguments. Collapse of density occurs at $t_* = 1 - C^{-1}$ on the surface in $\mathbb{R}^3(\boldsymbol{x})$ given by the equation

$$Cx + U(y/t_*, z/t_*) = 0. \tag{1.4}$$

Since U is an arbitrary function, Eq. (1.4) specifies a surface of a rather general form.

As noted above, solutions (1.1) and (1.3) are isentropic: $S = S_0 = \text{const}$, and the pressure is determined from the equation of state $p = F(\rho)$ with an arbitrary function F.

2. Example of a Barochronic Solution of Type (1,2). We consider the submodel generated by the subalgebra $L_5 = \{\partial_y, \partial_z, t\partial_x + \partial_u, t\partial_y + \partial_v, \partial_x + t\partial_z + \partial_w\}$. It has the invariants t, $w + tu - x$, ρ, and p

and the superfluous functions u and v. The solution is written as

$$u = u(t, \boldsymbol{x}), \qquad v = v(t, \boldsymbol{x}), \qquad w = x - tu.$$

The solution constructed from the initial data at $t = 0$ has the form

$$u_0 = u_0(\boldsymbol{x}), \qquad v_0 = v_0(\boldsymbol{x}), \qquad w_0 = x. \tag{2.1}$$

The case $t_0 \neq 0$ is reduced to (2.1) by substitution of the independent variables and velocities (the remaining variables remain unchanged):

$$Z = z + t_0 x, \qquad W = w + t_0 u. \tag{2.2}$$

Substitution (2.2) conserves the invariants of the matrix J_0.

The Jacobian matrix of the initial data (2.1) $J_0 = \partial \boldsymbol{u}_0 / \partial \boldsymbol{x}$ has the invariants

$$h_0 = u_{0x} + v_{0y}, \quad k_0 = u_{0x}v_{0y} - u_{0y}v_{0x} - u_{0z}, \quad m_0 = u_{0y}v_{0z} - u_{0z}v_{0y}. \tag{2.3}$$

System (2.3) is linearized by substitution of variables similarly to the canonical system for two-dimensional barochronic motion [3]. In contrast to the latter, which contains two functions of two independent variables, system (2.3) contains three equations for the two functions (u_0, v_0) of the three variables (x, y, z) and is overdetermined.

We use (x, z, u_0) as the new independent variables and (y, v_0) as the new functions, so that

$$y = Y(x, z, u_0), \qquad v_0 = V(x, z, u_0), \qquad Y_{u_0} \neq 0. \tag{2.4}$$

Differentiating expressions (2.4) over the variables (x, y, z), we obtain the following formulas for conservation of derivatives:

$$u_{0x} = -\frac{Y_x}{Y_u}, \;\; u_{0y} = \frac{1}{Y_{u_0}}, \;\; u_{0z} = \frac{Y_z}{Y_{u_0}}, v_{0x} = V_x - \frac{V_{u_0}Y_x}{Y_{u_0}}, \;\; v_{0y} = \frac{V_{u_0}}{Y_{u_0}}, \;\; v_{0z} = V_z - \frac{V_{u_0}Y_z}{Y_{u_0}}. \tag{2.5}$$

After substitution of expressions (2.5) into system (2.3), we obtain the following linear system for the functions Y and V:

$$V_{u_0} - Y_x = h_0 Y_{u_0}, \quad -V_x + Y_z = k_0 Y_{u_0}, \quad V_z = m_0 Y_{u_0}. \tag{2.6}$$

Cross differentiation eliminates the function V and yields an overdetermined system of three linear second-order equations for the function Y. It can be solved for three (of six) derivatives of Y to yield compatibility conditions of the next order. This cumbersome process can be eliminated by appropriate selection of new variables.

Let the matrix J_0 have three different real eigenvalues λ_{k0} ($k = 1, 2, 3$). The remaining cases are examined similarly.

We introduce new independent variables:

$$\alpha = u_0 - (\lambda_{10} + \lambda_{20})x + \lambda_{10}\lambda_{20}z, \qquad \beta = u_0 - (\lambda_{10} + \lambda_{30})x + \lambda_{10}\lambda_{30}z, \tag{2.7}$$

$$\gamma = u_0 - (\lambda_{20} + \lambda_{30})x + \lambda_{20}\lambda_{30}z.$$

The Jacobian of the transformation from the variables (u_0, x, z) to the variables (α, β, γ) is different from zero by virtue of the condition $\lambda_{i0} \neq \lambda_{j0}$ for $i \neq j$:

$$\Delta = \frac{\partial(\alpha, \beta, \gamma)}{\partial(u_0, x, z)} = (\lambda_{10} - \lambda_{20})(\lambda_{10} - \lambda_{30})(\lambda_{20} - \lambda_{30}) \neq 0.$$

We also introduce new functions:

$$P = V - \lambda_{30}Y, \qquad Q = V - \lambda_{20}Y, \qquad R = V - \lambda_{10}Y. \tag{2.8}$$

These functions are not independent. The condition of their linear dependence will be used below.

In the new variables (2.7) and (2.8), system (2.6) becomes

$$P_\alpha + Q_\beta + R_\gamma = 0, \qquad (\lambda_{10}+\lambda_{20})P_\alpha + (\lambda_{10}+\lambda_{30})Q_\beta + (\lambda_{20}+\lambda_{30})R_\gamma = 0,$$
$$\lambda_{10}\lambda_{20}P_\alpha + \lambda_{10}\lambda_{30}Q_\beta + \lambda_{20}\lambda_{30}R_\gamma = 0. \tag{2.9}$$

The discriminant of system (2.9), which is treated as a system of linear homogeneous algebraic equations for the variables P_α, Q_β, and R_γ, is equal to $\Delta \neq 0$. Hence, system (2.9) has only the trivial solution $P_\alpha = Q_\beta = R_\gamma = 0$.

Thus, according to the definition (2.8) of the quantities P, Q, and R, we have

$$V - \lambda_{30}Y = \varphi_1(\beta,\gamma), \quad V - \lambda_{20}Y = \varphi_2(\alpha,\gamma), \quad V - \lambda_{10}Y = \varphi_3(\alpha,\beta). \tag{2.10}$$

The functions P, Q, and R, and, hence, function (2.10) are related by one linear relation resulting from (2.8):

$$(\lambda_{10}-\lambda_{20})\varphi_1(\beta,\gamma) + (\lambda_{30}-\lambda_{10})\varphi_2(\alpha,\gamma) + (\lambda_{20}-\lambda_{30})\varphi_3(\alpha,\beta) = 0. \tag{2.11}$$

We differentiate Eq. (2.11) with respect to γ and β. Then we repeat this procedure twice, performing cyclic permutation of variables. As a result, we have

$$\frac{\partial^2\varphi_1}{\partial\beta\partial\gamma} = \frac{\partial^2\varphi_2}{\partial\alpha\partial\gamma} = \frac{\partial^2\varphi_3}{\partial\alpha\partial\beta} = 0.$$

Thus,

$$\varphi_1 = \varphi_1^1(\beta) + \varphi_1^2(\gamma), \quad \varphi_2 = \varphi_2^1(\alpha) + \varphi_2^2(\gamma), \quad \varphi_3 = \varphi_3^1(\alpha) + \varphi_3^2(\beta). \tag{2.12}$$

We substitute (2.12) into Eq. (2.11) and find that the sum of the three functions, each of which depends on its argument, is equal to zero. Hence, the variables are separated, and the functions φ_i^j $(i, j = 1, 2, 3)$ are related by

$$\begin{aligned}
(\lambda_{30}-\lambda_{10})\varphi_2^1(\alpha) + (\lambda_{20}-\lambda_{30})\varphi_3^1(\alpha) &= b - a,\\
(\lambda_{10}-\lambda_{20})\varphi_1^1(\beta) + (\lambda_{20}-\lambda_{30})\varphi_3^2(\beta) &= a - c,\\
(\lambda_{10}-\lambda_{20})\varphi_1^2(\gamma) + (\lambda_{30}-\lambda_{10})\varphi_2^2(\gamma) &= c - b,
\end{aligned} \tag{2.13}$$

where a, b, and c are arbitrary numbers ("constants of separation").

We revert to system (2.10). We separate from it the functions Y and V. For this, we obtain two linearly independent equations from (2.10). The first of them is the sum of Eqs. (2.10), and the second is their linear combination: in (2.10), the first equation is multiplied by λ_{20}, the second by λ_{10}, and the third by λ_{30}. Then the required system becomes

$$3V - h_0Y = \varphi_1 + \varphi_2 + \varphi_3, \qquad h_0V - k_0Y = \lambda_{20}\varphi_1 + \lambda_{10}\varphi_2 + \lambda_{30}\varphi_3. \tag{2.14}$$

Here h_0 and k_0 are the first and second invariants of the matrix J_0. We consider the regular case where the discriminant of system (2.14) is $d = h_0^2 - 3k_0 \neq 0$. Since the functions φ_k are arbitrary, the discriminant is assumed to be equal to unity, and φ_k is replaced by the new functions $d^{-1}\varphi_k$.

Under this condition, the solutions of system (2.14) have the form

$$Y = (2\lambda_{20}-\lambda_{10}-\lambda_{30})\varphi_1 + (2\lambda_{10}-\lambda_{20}-\lambda_{30})\varphi_2 + (2\lambda_{30}-\lambda_{10}-\lambda_{20})\varphi_3, \tag{2.15a}$$

$$V = (\lambda_{20}^2-\lambda_{10}\lambda_{30})\varphi_1 + (\lambda_{10}^2-\lambda_{20}\lambda_{30})\varphi_2 + (\lambda_{30}^2-\lambda_{10}\lambda_{20})\varphi_3. \tag{2.15b}$$

On the right side of formulas (2.15), we separate terms that depend only on the arguments α, β, and γ, respectively.

Let us show that solution (2.15) depends only on three functions, each of which depends on its argument. For this, we use relations (2.13) and the fact that, in view of the homogeneity of the initial system (2.6), its solutions Y and V (2.15) are determined with accuracy up to additive constants.

In formula (2.15a), we separate the term that depends only on α:

$$(2\lambda_{10}-\lambda_{20}-\lambda_{30})\varphi_2^1(\alpha) + (2\lambda_{30}-\lambda_{10}-\lambda_{20})\varphi_3^1(\alpha)$$

$$= (\lambda_{10} - \lambda_{20})\varphi_2^1(\alpha) + (\lambda_{30} - \lambda_{10})\varphi_3^1(\alpha) - [(\lambda_{30} - \lambda_{10})\varphi_2^1(\alpha) + (\lambda_{20} - \lambda_{30})\varphi_3^1(\alpha)]. \tag{2.16}$$

By virtue of the first equation of (2.13), the expression in square brackets in (2.16) is equal to the constant number $b - a$ and, hence, without loss of generality, it can be omitted in the solution formula. Thus, in formula (2.15a), the term depending on α has the form

$$f_1(\alpha) = (\lambda_{10} - \lambda_{20})\varphi_2^1(\alpha) + (\lambda_{30} - \lambda_{10})\varphi_3^1(\alpha). \tag{2.17}$$

In formula (2.15b), we separate the solution component that depends only on α and divide it into two terms:

$$\begin{aligned}(\lambda_{10}^2 - \lambda_{20}\lambda_{30})\varphi_2^1(\alpha) + (\lambda_{30}^2 - \lambda_{10}\lambda_{20})\varphi_3^1(\alpha) &= \lambda_3[(\lambda_{10} - \lambda_{20})\varphi_2^1(\alpha) + (\lambda_{30} - \lambda_{10})\varphi_3^1(\alpha)] \\ - \lambda_1[(\lambda_{30} - \lambda_{10})\varphi_2^1(\alpha) + (\lambda_{20} - \lambda_{30})\varphi_3^1(\alpha)] &= \lambda_{30} f_1(\alpha) - \lambda_{10}(b - a). \end{aligned} \tag{2.18}$$

Here the last equality is valid by virtue of (2.17) and the first equation of (2.13). Since Y and V are determined with accuracy up to additive constants, the constant term in (2.18) can be dropped.

Similar formulas hold for the solution components V and Y, which depend only on β and γ. Thus, the following lemma is valid.

Lemma. *The general solution of system* (2.16) *has the form*

$$Y = f_1(\alpha) + f_2(\beta) + f_3(\gamma), \qquad V = \lambda_{30} f_1(\alpha) + \lambda_{20} f_2(\beta) + \lambda_{10} f_3(\gamma), \tag{2.19}$$

where f_i ($i = 1, 2, 3$) *are arbitrary functions of their arguments.*

The velocity components (u, v, w) as functions of all the variable are obtained from (2.4) by substituting expressions (2.19) into the right sides, erasing zeros at u_0 and v_0, and substituting $\boldsymbol{x} \to \boldsymbol{x} - t\boldsymbol{u}$. The solution (u, v) is determined implicitly from the system obtained.

3. Submodels of Nonbarochronic Form. According to [2], for the EGD with an equation of state of general form, there are three such RPIS of type (1,2) and fourteen RPIS of type (1,1). The first are generated by 5- and 6-dimensional subalgebras from ΘL_{11}, and the second are generated by 4-dimensional subalgebras from ΘL_{11}. All these subalgebras contain operators transforming time t.

RPIS of this form have the invariant independent variable λ, which is a function of only the initial independent variables $\lambda = \lambda(t, \boldsymbol{x})$, and $\nabla\lambda \neq 0$. The variables p, ρ, and S are also invariants of the subalgebras. In addition, for submodels with defect $\delta = 2$, one velocity component is expressed in terms of invariants, and for submodels with $\delta = 1$, two velocity components are expressed in terms of invariants. The remaining velocity components (two or one) are superfluous functions.

All these submodels have a common structure and are integrated by a unified scheme including the following stages:

Stage I. After introduction of an auxiliary function, the invariant subsystem reduces to one ordinary differential equation for this function. Below, it is called equation B. All invariant functions are restored from the solution of B by quadratures. Equation B is the equation of momenta for the invariant velocity component. For some submodels, this equation can be integrated to give the invariant component of the Bernoulli integral. The latter is the final relation between this component and the enthalpy of the gas. Since the invariant velocity component and the enthalpy are expressed in terms of the auxiliary function and its derivatives, after substitution of these expressions into equation B, the latter becomes an ordinary differential equation for the auxiliary function.

Stage II. The overdetermined system, or, more precisely, its passive part is similar to the system that describes two-dimensional barochronic gas flows and is integrated similarly to it [3].

(A) We consider the present scheme for the submodel of type (1,2) generated by the subalgebra $L_5 = \{\partial_y, \partial_z, t\partial_y + \partial_v, t\partial_z + \partial_w, y\partial_z - z\partial_y + v\partial_w - w\partial_v + \partial_t\}$. It has the invariants (x, u, ρ, p, S) and the superfluous functions (v, w).

The solution has the form

$$v = v(t, \boldsymbol{x}), \qquad w = w(t, \boldsymbol{x}), \qquad (u, \rho, p, S) \mid x.$$

The equations of the submodel are written as

$$uu' + \rho^{-1}p' = 0, \tag{3.1a}$$

$$v_t + uv_x + vv_y + wv_z = 0, \tag{3.1b}$$

$$w_t + uw_x + vw_y + ww_z = 0, \tag{3.1c}$$

$$u\rho' + \rho(u' + v_y + w_z) = 0, \tag{3.1d}$$

$$uS' = 0, \tag{3.1e}$$

where the prime denotes differentiation with respect to x. According to Eq. (3.1e), for the entropy, we have the following two possibilities.

1. Let $u = 0$. In this case, from (3.1a) we have $p = p_0 = \text{const}$. Then, system (3.1) describes two-dimensional isobaric gas flows [9]. The corresponding solution is written as

$$u = 0, \tag{3.2a}$$

$$z = (y - tv)f_1'(x, v) + f_2(x, v) + f_1(x, v), \tag{3.2b}$$

$$w = f_1(x, v), \tag{3.2c}$$

$$\rho = R(x, v, y - tv), \tag{3.2d}$$

$$S = H(x, v, y - tv), \tag{3.2e}$$

$$p_0 = F(\rho, S), \tag{3.2f}$$

where f_1, f_2, R, and H are arbitrary functions of their arguments. The last two are related by (3.2f), which is an isobaric equation of state. The velocity component v is specified by relation (3.2b) as an implicit function of the variables (t, x, y, z).

This solution is a double wave [9] since all required functions depend on the two Lagrangian invariants v and $y - tv$, which are functionally independent for the variables (y, z) at $v_z \neq 0$.

2. Let $u \neq 0$. Then, from (3.1e) we have $S = S_0 = \text{const}$. The submodel describes isentropic gas flow. We introduce the Lagrangian variable

$$\xi = t - \int u^{-1}\, dx.$$

Then Eqs. (3.1b)–(3.1d) become the standard system

$$v_t + vv_y + wv_z = 0, \quad w_t + vw_y + ww_z = 0, \quad v_y + w_z = h(x), \tag{3.3}$$

where

$$h = -(u(\ln\rho)' + u'). \tag{3.4}$$

System (3.3) is similar to the system describing two-dimensional barochronic gas flows [3] and is integrated similarly to it. The compatibility conditions for system (3.3) are the equations

$$Dh + h^2 = 2k, \qquad Dk + hk = 0, \tag{3.5}$$

where $h = \mathrm{sp}J$, $k = \det J$, and $D = u\partial_x$.

We introduce the new independent variable $X = X(x)$:

$$\frac{dX}{dx} = \frac{1}{u}, \tag{3.6}$$

so that $X = \int \frac{dx}{u(x)}$. Then, Eqs. (3.5) coincide with the conditions of compatibility of the equations of two-dimensional barochronic motion (X plays the role of t).

As is known [6], Eqs. (3.5) have the general solution

$$h = Q_X/Q, \tag{3.7a}$$

$$k = Q_{XX}/2Q, \tag{3.7b}$$

where $Q = 1 + h_0 X + k_0 X^2$, h_0, and $k_0 = \text{const}$.

The eigenvalues $\lambda_k = \lambda_k(x)$ $(k = 1)$ of the matrix J satisfy the equations $\lambda_{kX} + \lambda_k^2 = 0$ $(k = 1, 2)$, which have the solutions

$$\lambda_k = \frac{\lambda_{k0}}{1 + \lambda_{k0} X} \quad (\lambda_{k0} = \text{const}, \quad k = 1, 2). \tag{3.8}$$

In this case, $\lambda_1 + \lambda_2 = h$, $\lambda_1 \lambda_2 = k$ and $\lambda_{10} + \lambda_{20} = h_0$, $\lambda_{10}\lambda_{20} = k_0$. Depending on the sign of the discriminant $d_0 = h_0^2/4 - k_0$, we have the corresponding representation of the solution of system (3.3) [5].

Thus, if $d_0 > 0$, the matrix J_0 has two different real eigenvalues. In this case, the functions $(v, w)\big|(t, X, y, z)$ are determined implicitly from the system

$$F_k\left(y - \frac{1}{\lambda_k} v, z - \frac{1}{\lambda_k} w, t - X\right) = 0 \qquad (k = 1, 2) \tag{3.9}$$

for arbitrary functions F_k, such that their gradients for the first couple of variables are linearly independent. This condition guarantees local resolvability of system (3.9) for (v and w). Since Eqs. (3.9) can be solved for one of the arguments, the solution obtained depends on two arbitrary functions of two variables.

The remaining invariant subsystem, containing Eqs. (3.1a), (3.4), and (3.5) [or (3.7)], is reduced to equation B (3.1a) and quadratures. The continuity equation (3.4) in terms of the variable X is integrated in explicit form. Indeed,

$$u(\ln \rho)_x = \frac{d \ln \rho}{dX}, \quad u_x = \frac{1}{u}\frac{du}{dX} = \frac{d \ln u}{dX}.$$

Substituting these expressions and representation (3.7a) for h into (3.4) and integrating, we obtain $\rho = R_0/uQ$, where $R_0 = \text{const}$. Changing the roles of the dependent and independent variables, i.e., setting $x = x(X)$, we obtain the following explicit formula for density:

$$\rho = \frac{R_0}{(1 + h_0 X + k_0 X^2) x_X}. \tag{3.10}$$

Thus, all required functions are represented as functions of the variables (t, X, y, z). The component u is determined from (3.6), (v, w) is determined from (3.9), and ρ from (3.10).

To obtain the final solution, it remains to find the function $x = x(X)$. It satisfies the equation of momenta (3.1a), which is integrated once to take the form of the invariant component of the Bernoulli integral

$$\frac{1}{2}u^2 + I(\rho) = b_0, \qquad b_0 = \text{const}, \tag{3.11}$$

where the enthalpy is $I(\rho) = \int \rho^{-1}\, dp$. Equation B (3.11) into which u from (3.6) and ρ from (3.10) are substituted is an ordinary differential first-order equation for the function $x = x(X)$.

We write it in expanded form for a gas with a polytropic equation of state $p = \rho^\gamma$:

$$(1 + h_0 X + k_0 X^2)^{\gamma-1}(x_X^{\gamma+1} - 2b_0 x_X^{\gamma-1}) + æ = 0, \tag{3.12}$$

where $æ = 2\gamma R_0^{\gamma-1}/(\gamma - 1) = \text{const}$. For some values of γ, Eq. (3.12) can be integrated in elementary functions.

(B) We describe integration of the equations of the submodel of type (1,1) generated by the subalgebra $L_4 = \{\partial_y, \partial_z, y\partial_z - z\partial_y + v\partial_w - w\partial_v, t\partial_x + \partial_u + \partial_t\}$. For this, it is convenient to introduce polar coordinates in the hodograph plane (v, w): $v = q\cos\varphi$ and $w = q\sin\varphi$.

The subalgebra has the invariants $\lambda = x - t^2/2$, $u - t$, q, ρ, S, and p and the superfluous function φ. The solution has the form

$$u = t + U(\lambda), \quad v = q(\lambda)\cos\varphi, \quad w = q(\lambda)\sin\varphi, \quad (\rho, S, p)\Big|\lambda, \quad \varphi = \varphi(t, x, y, z).$$

The submodel is defined by the equations

$$UU' + 1 + \rho^{-1}p' = 0, \tag{3.13a}$$

$$\varphi_t + U\varphi_\lambda + q(\varphi_y\cos\varphi + \varphi_z\sin\varphi) = 0, \tag{3.13b}$$

$$Uq' = 0, \tag{3.13c}$$

$$U\rho' + \rho(U' + q(-\varphi_y\sin\varphi + \varphi_z\cos\varphi)) = 0, \tag{3.13d}$$

$$US' = 0, \tag{3.13e}$$

where the prime denotes differentiation with respect to λ.

1. Let $U = 0$. Then, system (3.13) is reduced to a standard system similar to the system for helical barochronic gas flows with zero divergence. It is integrated to take the finite form $u = t$, $v = q\cos\varphi$, $w = q\sin\varphi$, $(\rho, S, p) \mid \lambda$, where the functions ρ, S, and p satisfy the equation of state $p = F(\rho, S)$ and the relation $p' + \rho = 0$. The function $q = q(\lambda)$ is arbitrary, and the function $\varphi = \varphi(t, \lambda, y, z)$ is defined implicitly by the equation $y\cos\varphi + z\sin\varphi = f(\lambda, \varphi) + tq$ for an arbitrary function f of its arguments.

2. Let $U \neq 0$. From (3.13e) it follows that $S = S_0 = \text{const}$, and the submodel describes isentropic gas flows. From Eq. (3.13c) it follows that $q = q_0 = \text{const}$.

We introduce the Lagrangian variable $\xi = t - \int U^{-1}\,d\lambda$. Then Eqs. (3.13b) and (3.13d) form a standard system similar to the system describing helical barochronic flows:

$$\varphi_t + q_0(\varphi_y\cos\varphi + \varphi_z\sin\varphi) = 0, \quad -\varphi_y\sin\varphi + \varphi_z\cos\varphi = h/q_0, \tag{3.14}$$

where

$$h = -[U(\ln\rho)' + U']. \tag{3.15}$$

The compatibility conditions for system (3.14) have the form

$$Uh' + h^2 = 0. \tag{3.16}$$

(a) Let $h \neq \text{const}$. We introduce the function $\sigma = \sigma(\lambda)$, so that $\sigma = 1/h$. Then, (3.16) leads to the representation

$$U = \frac{1}{\sigma'}, \qquad h = \frac{1}{\sigma}. \tag{3.17}$$

The continuity equation (3.15) with allowance for (3.17) is integrated in finite form. We obtain the following representation for the solution:

$$u = t + 1/\sigma', \quad v = q_0\cos\varphi, \quad w = q_0\sin\varphi, \quad \rho = R_0\sigma'/\sigma, \qquad S = S_0, \quad p = F(\rho), \tag{3.18}$$

where q_0, R_0, and $S_0 = \text{const}$. The function $\varphi = \varphi(t, \lambda, y, z)$ in (3.18) is defined implicitly by the relation $\Phi(\xi, y - q_0\sigma\cos\varphi, z - q_0\sigma\sin\varphi) = 0$ for an arbitrary function Φ.

The auxiliary function $\sigma = \sigma(\lambda)$ is found from equation B, which is the invariant component of the Bernoulli integral and is obtained by integration of the equation of momenta (3.13a):

$$\frac{1}{2}U^2 + \lambda + I(\rho) = b_0, \qquad b_0 = \text{const}. \tag{3.19}$$

Into Eq. (3.19) one should substitute expressions (3.17) of U and ρ in terms of σ and σ'. For a gas with a polytropic equation of state $p = \rho^\gamma$ Eq. (3.19) has the form

$$æ\sigma'^{\gamma+1} + 2(\lambda - b_0)\sigma^{\gamma-1}\sigma'^2 + \sigma^{\gamma-1} = 0,$$

where $æ = 2\gamma R_0^{\gamma-1}/(\gamma - 1) = \text{const}$.

(b) Let $h = h_0 = \text{const}$. Then, from Eq. (3.16) it follows that $h_0 = 0$. In this case, the functions $U = U(\lambda)$ and $\rho = \rho(\lambda)$ are determined from the consumption and Bernoulli integrals:

$$\rho U = Q_0, \qquad \frac{1}{2}U^2 + \lambda + I(\rho) = b_0, \qquad b_0 = \text{const}. \tag{3.20}$$

The function $\varphi = \varphi(\lambda, \xi, y, z)$ is found implicitly from the equation $y \cos\varphi + z \sin\varphi = f(\xi, \varphi) + tq_0$ with an arbitrary function f of its arguments. The variable ξ is determined from the $U = U(\lambda)$ obtained from (3.20).

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REFERENCES

1. L. V. Ovsyannikov, *Group Analysis of Differential Equations* [in Russian], Nauka, Moscow (1978).
2. L. V. Ovsyannikov, "Regular and irregular partially invariant solutions," *Dokl. Ross. Akad. Nauk*, **343**, No. 2, 156–159 (1995).
3. L. V. Ovsyannikov and A. P. Chupakhin, "Regular partially invariant submodels of gas-dynamic equations," *Prikl. Mat. Mekh.*, **60**, No. 6, 990–999 (1996).
4. L. V. Ovsyannikov, "Type (2,1) regular submodels of the equations of gas dynamics," *Prikl. Mekh. Tekh. Fiz.*, **37**, No. 2, 3–13 (1996).
5. A. P. Chupakhin, "Barochronic gas flows," *Dokl. Ross. Akad. Nauk*, **352**, No. 5, 624–626 (1997).
6. A. P. Chupakhin, "Submodel of barochronic gas motions," in: *Proc. Int. Conf. Modern Group Analysis VI* (Johannesburg, South Africa, January 15–20, 1996), New Delhi–New Age (1997), pp. 55–64.
7. Yu. G. Rykov, "Variational principle for a two-dimensional system of gas-dynamic equations without pressure," *Usp. Mat. Nauk*, **51**, No. 1, 165–166 (1996).
8. E. Weinan, Yu. G. Rykov, and Ya. G. Sinai, "Generalized variational principles, global weak solutions, and behavior with random initial data for system of conservation laws arising in adhesion particle dynamics," *Commun. Math. Phys.*, **177**, 349–380 (1996).
9. L. V. Ovsyannikov, "Isobaric gas flows," *Differ. Uravn.*, **30**, No. 10, 1792–1799 (1994).